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Effective Field Theory Approach to High-Temperature Thermodynamics

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Abstract

An effective field theory approach is developed for calculating the thermodynamic properties of a field theory at high temperature T and weak coupling g. The effective theory is the 3-dimensional field theory obtained by dimensional reduction to the bosonic zero-frequency modes. The parameters of the effective theory can be calculated as perturbation series in the running coupling constant $g^2(T)$. The free energy is separated into the contributions from the momentum scales T and gT, respectively. The first term can be written as a perturbation series in $g^2(T)$. If all forces are screened at the scale gT, the second term can be calculated as a perturbation series in $g(T)$ beginning at order g^3. The parameters of the effective theory satisfy renormalization group equations that can be used to sum up leading logarithms of $T/(gT)$. We apply this method to a massless scalar field with a Φ^4 interaction, calculating the free energy to order $g^6 \log g$ and the screening mass to order $g^5 \log g$.

I. Introduction

When matter is subjected to sufficiently extreme temperature or density, both quantum and relativistic effects become important. Such conditions arise in astrophysics and in cosmology and it may be possible to create them experimentally using heavy-ion collisions. A system at such an extreme temperature or density is most appropriately described by quantum field theory. The thermodynamic functions that describe the bulk equilibrium properties of such a system are given by the free energy density and its derivatives. If the temperature T is high enough that all masses can be neglected, the free energy density depends only on T and on the coupling constants of the field theory.

In recent years, there have been significant advances in the perturbative calculation of the free energy for high temperature field theories. The free energy for a massless scalar field with a Φ^4 interaction was computed to order g^4 by Frenkel, Saa, and Taylor [1] in 1992, and the order-g^5 correction was recently calculated by Parwani and Singh [2]. The free energy for high temperature QED was calculated to order e^4 by Coriano and Parwani [3], and extended to order e^5 by Parwani [4]. The free energy for a quark-gluon plasma in the high temperature limit was recently calculated to order g^4 by Arnold and Zhai [5]. The importance of these calculations goes far beyond simply determining one more term in the perturbation series. The leading term in the perturbation series is just the free energy of an ideal gas. The order-g^2 correction takes into account interactions between particles in the ideal gas. At order g^3, there is a qualitatively new contribution to the free energy. If the temperature is large compared to the masses of the particles, the force mediated by the exchange of a particle is long-range compared to the typical separation of the particles, which is of order $1/T$. The system therefore behaves like a plasma, screening the long-range interaction beyond the scale $1/(gT)$. It is this screening that is responsible for the g^3 term in the free energy density. Because of renormalization effects, the corrections of order g^4 and g^5 are also important. The coupling constant $g(\mu)$ depends on an arbitrary renormalization scale μ. The resulting ambiguity in the leading nontrivial term in a perturbative expansion can only be reduced

by a next-to-leading order calculation. While it may be clear on physical grounds that the scale μ should be of order T, the difference between the choices $g(T)$ and $g(2\pi T)$ can be of great practical significance. Thus a calculation to order g^4 is needed in order to determine the appropriate scale μ in the order-g^2 correction to the ideal gas term, while a calculation to order g^5 is required in order to determine the scale in the order-g^3 plasma term.

In the case of QCD, there is also a qualitatively new effect that arises at order g^6. As pointed out by Linde [6, 7] in 1979, the loop expansion for the free energy breaks down at this order in g. Up to order g^5, the free energy can be calculated using a resummation of perturbation theory that takes into account the screening of the chromoelectric force at distances of order $1/(gT)$. However the chromomagnetic force is not screened at the scale gT, and this causes a breakdown in the resummed perturbation expansion at order g^6. This breakdown has been widely interpreted as implying that nonperturbative effects enter in at this order, rendering the perturbation series meaningless beyond order g^5. This longstanding problem was recently solved by constructing a sequence of two effective field theories that are equivalent to thermal QCD over successively longer length scales [8]. The first effective theory reproduces the static gauge-invariant correlators of thermal QCD at distances of order $1/(gT)$ or larger, while the second effective theory reproduces the correlators at distances of order $1/(g^2T)$ or larger. Using this construction, the free energy can be separated into contributions from the momentum scales T, gT, and g^2T, with well-defined weak coupling expansions that begin at order g^0, g^3, and g^6, respectively. The contributions from the scales T and gT can be computed using perturbative methods. For the contribution from the scale g^2T, the coefficients in the weak-coupling expansion can be calculated using lattice simulations of pure-gauge QCD in 3 Euclidean dimensions. This result demonstrates the power of the effective-field-theory approach. This power can also be brought to bear on other fundamental problems in thermal field theory. For example, it has also been used to determine the correct asymptotic behavior of the correlator of Polyakov loop operators in high temperature QCD [9].

Effective field theory provides an effective method for unravelling the effects of the various momentum scales that arise in field theory at high temperature. In addition to providing insight into the qualitative behavior of the theory, effective field theory can also be used to streamline perturbative calculations, such as those in Refs. [1]-[5]. In this paper, we use effective field theory to develop a practical method for calculating the thermodynamic functions of a field theory at high temperature T and weak coupling g. To illustrate the method, we apply it to a massless scalar field theory with a Φ^4 interaction. In section II, we explain how the effective field theory that describes distance scales of order $1/(gT)$ or larger is related to dimensional reduction to the bosonic zero-frequency modes. In Section III, we show how the parameters of the effective theory can be obtained by matching perturbative calculations in the effective theory and in the full theory. In Section IV, we use the effective theory for the massless Φ^4 theory to calculate the free energy to order g^5 and the screening mass to order g^4. The accuracy of these calculations is improved in Section V to order $g^6 \log g$ for the free energy and to order $g^5 \log g$ for the screening mass by using renormalization group equations for the parameters of the effective theory. We conclude in Section VI with a discussion of the application of our effective-field-theory approach to gauge theories and with a brief comparison with related work. In appendices A and B, we collect all the necessary formulas for sum-integrals in the full theory and for integrals in the 3-dimensional effective theory. In appendix C, we derive the renormalization group equations for the parameters of the effective theory.

II. Dimensional Reduction

In the limit of high temperature T, the static correlation functions of a field theory in (3+1) dimensions can be reproduced at long distances $R \gg 1/T$ by an effective field theory in 3 dimensions. This idea, which is called "dimensional reduction", has a long history [7, 10, 11, 12]. It has provided insight into the qualitative behavior of the field theory at high temperature, but it has never been fully exploited for quantitative calculations.

Dimensional reduction is based on the fact that static correlation functions for a field theory in thermal equilibrium can be expressed in terms of Euclidean functional integrals. The partition function $\mathcal{Z}$ is defined by

$$\mathcal{Z}(T) \;=\; \mathrm{trace}\left(e^{-\beta H}\right), \tag{1}$$

where H is the hamiltonian operator and $\beta = 1/T$. The operator $e^{-\beta H}$ is the evolution operator that evolves a state from time $t = 0$ to the imaginary time $t = -i\beta$. It can be represented as a functional integral over fields $\Phi(\mathbf{x}, t)$ defined on the time interval from 0 to $-i\beta$. It is natural to change variables from t to the imaginary time $\tau = it$. The partition function (1) is then given by the Euclidean functional integral

$$\mathcal{Z}(T) \;=\; \int \mathcal{D}\Phi(\mathbf{x},\tau)\; \exp\left(-\int_0^\beta d\tau \int d^3x\; \mathcal{L}\right), \tag{2}$$

where $\mathcal{L}$ is the negative of the lagrangian density for the (3+1)-dimensional theory with the time t analytically continued to $-i\tau$. The trace in (1) is implemented by imposing boundary conditions on the fields:

$$\Phi(\mathbf{x}, \tau = \beta) \;=\; \pm\, \Phi(\mathbf{x}, \tau = 0), \tag{3}$$

where the plus sign holds for bosonic fields and the minus sign for fermionic fields. The correlator of two operators $\mathcal{O}(\mathbf{0})$ and $\mathcal{O}(\mathbf{R})$ is obtained by averaging their product over fields $\Phi(\mathbf{x}, \tau)$ with the exponential weighting factor in (2).

Because of the periodicity conditions (3) on the fields, they can be decomposed into Fourier modes in τ, with Matsubara frequencies $\omega_n = 2n\pi T$ for bosons and $\omega_n = (2n+1)\pi T$ for fermions. The contribution to a correlator from the exchange of a Fourier mode with frequency ω_n falls off at large R like $\exp(-|\omega_n|R)$. Thus the only modes whose contributions do not fall off exponentially at distances greater than $1/T$ are the $n = 0$ modes of the bosons. This suggests the strategy of integrating out the fermionic modes and the nonzero modes of the bosons to get an effective theory for the bosonic zero modes. This process is called "dimensional reduction". It results in a 3-dimensional Euclidean field theory with

bosonic fields only which reproduces the static correlators of the original theory at distances $R \gg 1/T$.

Constructing the dimensionally-reduced effective theory by actually integrating out degrees of freedom is cumbersome beyond leading order in the coupling constant. Once the appropriate 3-dimensional fields and their symmetries have been identified, a better strategy is to use the methods of "effective field theory" [13]. One writes down the most general lagrangian $\mathcal{L}_{\rm eff}$ for the 3-dimensional fields that respects the symmetries. This effective lagrangian has infinitely many parameters, but they are not arbitrary. By computing static correlators in the full theory, computing the corresponding correlators in the effective theory, and demanding that they agree at distances $R \gg 1/T$, one can determine the parameters of $\mathcal{L}_{\rm eff}$ in terms of T and the parameters of the original theory. Note that this matching procedure does not necessarily require the explicit determination of the relation between the fields in the effective theory and the fundamental fields.

The construction of the 3-dimensional effective theory is complicated by ultraviolet divergences. The ultraviolet divergences associated with the original 4-dimensional theory are removed by the standard renormalization procedure. However the 3-dimensional effective theory also has ultraviolet divergences whose origin can be traced to integrating out the nonzero-frequency modes in the full theory. They must be regularized by introducing an ultraviolet cutoff Λ. The parameters in $\mathcal{L}_{\rm eff}$ must therefore depend on Λ in such a way as to cancel the Λ-dependence of the regularized loop integrals in the effective theory. The natural scale for the ultraviolet cutoff Λ is of order T, since this is the scale at which the corresponding integrals are cut off by the nonzero modes in the full theory. It is useful, however, to keep the cutoff Λ independent of T, so that the only dependence on T in the effective theory comes from the parameters in $\mathcal{L}_{\rm eff}$. From the point of view of the full theory, the ultraviolet cutoff Λ of the effective theory plays the role of an arbitrary factorization scale that is introduced to separate the momentum scale T from lower momentum scales, such as gT, which can be described within the effective theory.

The ultraviolet divergences of the effective theory include power ultraviolet divergences of the form Λ^p, $p = 1, 2, 3, \ldots$, and logarithmic ultraviolet divergences of the form $\log(\Lambda/m)$, where m is a mass scale in the effective theory. The power divergences and the logarithmic divergences are quite different in character. The coefficients of the power divergences depend on the regularization procedure and are therefore simply regularization artifacts. In particular, the coefficient of Λ^p in some observable calculated in the effective theory is independent of the coefficient of T^p for that observable in the full theory. Therefore power divergences of the form Λ^p from loop integrals must be completely cancelled by terms proportional to Λ^p in the parameters of $\mathcal{L}_{\rm eff}$. In contrast to the power divergences, logarithmic ultraviolet divergences have coefficients that are independent of the regularization procedure. The reason for this is that a logarithmic ultraviolet divergence of the form $\log(\Lambda/m)$ from a loop integral must match onto a $\log(T/\Lambda)$ term in one of the parameters of $\mathcal{L}_{\rm eff}$ in order for the Λ-dependence to cancel. Thus the logarithmic divergences in the effective theory are related to logarithms of T in the full theory and therefore have real physical significance. Because of the unphysical character of power ultraviolet divergences, it is convenient to use a regularization procedure for the effective theory in which power ultraviolet divergences are subtracted and the remaining logarithmically divergent integrals are cut off at the scale Λ. With such a regularization procedure, only the logarithmic divergences need to be cancelled by the parameters of the effective theory. The subtraction of power divergences can be implemented with any cutoff procedure, including a momentum cutoff or lattice regularization. For perturbative calculations, a particularly convenient cutoff procedure is dimensional regularization, in which momentum integrals are analytically continued to $3 - 2\epsilon$ spacial dimensions. Power divergences are automatically subtracted with this method, because integrals without any momentum scale are 0 by definition in dimensional regularization. The remaining logarithmic ultraviolet divergences appear as poles in ϵ, and the renormalization procedure can be completed by subtracting these poles. If Λ is the momentum scale introduced by dimensional regularization, then this "minimal subtraction" procedure is equivalent to cutting off

the logarithmic ultraviolet divergences at a momentum scale of order Λ.

We turn now to the specific case of a massless scalar field with a Φ^4 interaction. In the partition function (2), the Euclidean lagrangian density is

$$\mathcal{L} \;=\; \frac{1}{2}\left(\partial_\tau \Phi\right)^2 \;+\; \frac{1}{2}\left(\boldsymbol{\nabla}\Phi\right)^2 \;+\; \frac{1}{4!} g^2 \Phi^4. \tag{4}$$

The 3-dimensional effective field theory obtained by dimensional reduction describes a scalar field $\phi(\mathbf{x})$ that can be approximately identified with the zero-frequency mode of the field in the original theory:

$$\sqrt{T} \int_0^\beta d\tau \; \Phi(\mathbf{x}, \tau) \;\approx\; \phi(\mathbf{x}). \tag{5}$$

The symmetries of the effective theory are rotational symmetry and the discrete symmetry $\phi(\mathbf{x}) \rightarrow -\phi(\mathbf{x})$, which follows from the symmetry $\Phi \rightarrow -\Phi$ of the fundamental lagrangian (4). The effective lagrangian has the general form

$$\mathcal{L}_{\rm eff} \;=\; \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 \;+\; \frac{1}{2} m^2(\Lambda)\, \phi^2 \;+\; \frac{1}{4!}\lambda(\Lambda)\, \phi^4 \;+\; \delta\mathcal{L}, \tag{6}$$

where $\delta\mathcal{L}$ includes all other local terms that are consistent with the symmetries. The parameters $m^2(\Lambda)$ and $\lambda(\Lambda)$ and the infinitely many parameters in (6) depend on the ultraviolet cutoff Λ, the temperature T, and the coupling constant g^2. The partition function (2) can be expressed as a Euclidean functional integral over the 3-dimensional field $\phi(\mathbf{x})$:

$$\mathcal{Z}(T) \;=\; e^{-f(\Lambda)\, V} \int^{(\Lambda)} \mathcal{D}\phi(\mathbf{x}) \exp\left(-\int d^3x \; \mathcal{L}_{\rm eff}\right). \tag{7}$$

The parameter $f(\Lambda)$ in the exponential prefactor is the coefficient of the unit operator in the effective lagrangian, which was omitted from $\mathcal{L}_{\rm eff}$ in (6). In addition to depending on g^2 and T, it also depends on Λ in such a way as to cancel the Λ-dependence of the functional integral in (7). The correlator of operators $\mathcal{O}(\mathbf{0})$ and $\mathcal{O}(\mathbf{R})$ in the full theory can be calculated at long distances $R \gg 1/T$ by identifying the corresponding operators in the effective theory and averaging their product over fields $\phi(\mathbf{x})$ with the exponential weighting factor in (7).

Renormalization theory implies that correlators at long distances $R \gg 1/T$ can be reproduced to any desired accuracy by adding sufficiently many operators to the effective

lagrangian and tuning their coefficients with sufficient accuracy as functions of g^2, T, and Λ [13]. With the three terms in the lagrangian that are given explicitly in (6), long-distance correlators can only be reproduced with limited accuracy. If we include the operator ϕ^6, the field theory becomes renormalizable, but this renormalizable theory is still only accurate up to a finite order in the coupling constant g. This has been interpreted as a breakdown of dimensional reduction [12], but the correct interpretation is simply that nonrenormalizable operators must also be included in $\mathcal{L}_{\rm eff}$ in order to extend the accuracy to higher order in g. In general, one must include all operators that are invariant under rotations and under the symmetry $\phi \to -\phi$. The resulting field theory is nonrenormalizable and has infinitely many parameters. These parameters, however, are not arbitrary, but are determined as functions of g^2, T, and Λ by the condition that the effective theory reproduce the long distance behavior of the original theory.

It is easy to determine the magnitude of the coefficient of a general operator in the effective lagrangian. From the kinetic term $(\boldsymbol{\nabla}\phi)^2$ in (6), we see that the field ϕ should be assigned a scaling dimension of 1/2. The operators given explicitly in (6) then have dimensions 3, 1, and 2. If we use a renormalization procedure in which power ultraviolet divergences are subtracted, then by dimensional analysis, an operator of dimension d in the effective lagrangian must have a coefficient that is proportional to T^{3-d}. It remains only to determine its order in g. The operator ϕ^4 is generated at tree level from the $g^2\Phi^4$ term in the original lagrangian, and therefore has a coefficient proportional to g^2. All other interaction terms in $\mathcal{L}_{\rm eff}$ arise from loop diagrams in the effective theory. Operators with $2n$ powers of ϕ are generated by 1-loop diagrams with n 4-point interactions and therefore have coefficients of order g^{2n}. Thus an operator with the schematic structure $\nabla^{2m}\phi^{2n}$ has dimension $d = 2m+n$ and will appear in $\mathcal{L}_{\rm eff}$ with a coefficient of magnitude $g^{2n}T^{3-d}$. The case $n = 1$ is an exception, because the 1-loop diagram with two external lines is momentum-independent and therefore only the operator ϕ^2 is generated at order g^2. Operators of dimension $d = 2m+1$ with the schematic structure $\nabla^{2m}\phi^2$ have coefficients with magnitude g^4T^{3-d} for $m \geq 2$. The

only other exception is ϕ^4, which is generated at tree level and has a coefficient of magnitude g^2T.

III. Short-distance Coefficients

The coefficients of the operators in the effective lagrangian (6) must be tuned as functions of g, T, and Λ so that the effective theory reproduces the static correlation functions of the full theory at distances $R \gg 1/T$. The parameters can be determined by computing various static quantities in the full theory, computing the corresponding quantities in the effective theory, and demanding that they match. If the running coupling constant $g^2(T)$ of the full theory is small at the scale T, then it is convenient to carry out these calculation using perturbation theory in $g^2(T)$. A strict perturbation expansion in g^2 is afflicted with infrared divergences due to long-range forces mediated by massless particles. These divergences are screened at the scale gT, but this screening can only be taken into account by summing up infinite sets of higher-order diagrams. This breakdown of perturbation theory does not prevent its use as a device for determining the short-distance coefficients in the effective lagrangian. As long as we can carry out perturbative calculations in the effective theory that make the same incorrect assumptions about the long-distance behavior as perturbation theory in the full theory, we can match the results and determine the short-distance coefficients.

In the case of the Φ^4 theory, conventional perturbation theory in g^2 corresponds to decomposing the lagrangian (4) as $\mathcal{L} = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int}$, where

$$\begin{aligned} \mathcal{L}_{\rm free} &= \frac{1}{2}(\partial_\tau \Phi)^2 \; + \; \frac{1}{2}(\boldsymbol{\nabla}\Phi)^2, \\ \mathcal{L}_{\rm int} &= \frac{g^2}{4!}\Phi^4. \end{aligned} \tag{8}$$

We will refer to the resulting perturbation theory as a "strict" perturbation expansion in g^2. The free part of the lagrangian describes a massless scalar field. A mass will not be generated at any finite order in g^2, and the absence of a mass will give rise to infrared divergences that

become more and more severe as you go to higher and higher orders in g^2. This behavior is physically incorrect. It will be clear from the effective theory that a mass m of order gT is generated by higher loop diagrams and it provides the screening that cuts off the infrared divergences. One way of dealing with the infrared divergences in the full theory is to use a reorganization of perturbation theory that incorporates the effects of the mass m in the free part of the lagrangian. The simplest possibility [15] is to write the lagrangian (4) as $\mathcal{L} = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int}$, where

$$\begin{aligned} \mathcal{L}_{\rm free} &= \frac{1}{2}(\partial_\tau \Phi)^2 + \frac{1}{2}(\boldsymbol{\nabla}\Phi)^2 + \frac{1}{2}m^2\Phi^2, \\ \mathcal{L}_{\rm int} &= \frac{g^2}{4!}\Phi^4 - \frac{1}{2}m^2\Phi^2. \end{aligned} \tag{9}$$

Both g^2 and m^2 in the interaction term are treated as perturbation parameters of the same order. The mass parameter m^2 must be of order g^2T^2 in order to avoid large perturbative corrections that grow quadratically with T in the high temperature limit. Another possibility is to add the mass term to $\mathcal{L}_{\rm free}$ and subtract it from $\mathcal{L}_{\rm int}$ only for the zero-frequency mode of $\Phi(\mathbf{x}, \tau)$ [14]. Both of these approaches have the drawback that the resulting sums and integrals involve two momentum scales T and m, making calculations unnecessarily difficult.

A simpler approach is to calculate in both the full theory and the effective theory using ordinary perturbation theory in g^2, but with an infrared cutoff to regularize the infrared divergences. In the full theory, this strict perturbation expansion in g^2 is defined by the decomposition (8). In the effective lagrangian (6), the coefficients m^2 and λ are of order g^2 and all the coefficients in $\delta\mathcal{L}$ are of order g^4 or higher. Thus, in the effective theory, the strict expansion in g^2 is defined by the decomposition $\mathcal{L}_{\rm eff} = (\mathcal{L}_{\rm eff})_{\rm free} + (\mathcal{L}_{\rm eff})_{\rm int}$, where

$$\begin{aligned} (\mathcal{L}_{\rm eff})_{\rm free} &= \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2, \\ (\mathcal{L}_{\rm eff})_{\rm int} &= \frac{1}{2}m^2(\Lambda)\,\phi^2 + \frac{1}{4!}\lambda(\Lambda)\,\phi^4 + \delta\mathcal{L}. \end{aligned} \tag{10}$$

The expansions in g^2 defined by (8) and (10) both generate infrared divergences that become more and more severe in higher orders. But if the parameters in the effective lagrangian are

tuned so that the two theories are equivalent at long distances, then the infrared divergences in their strict perturbative expansions will also match. Thus, in spite of the fact that the strict expansion in g^2 gives a physically incorrect treatment of infrared effects, we can use it as a device for computing short-distance coefficients.

IIIa. Coefficient of the unit operator

In this subsection, we calculate the parameter f in (7) to next-to-next-to-leading order in g^2. The parameter f can be interpreted as the coefficient of the unit operator which has been omitted from the effective lagrangian (6). We will determine f by matching calculations of $\log \mathcal{Z}$ in the full theory and in the effective theory.

We first calculate $\log \mathcal{Z}$ to next-to-next-to-leading order in g^2 using the perturbation expansion for the full theory defined by the decomposition (8). It is given by the sum of the Feynman diagrams in Fig. 1:

$$\begin{aligned} \frac{T\ \log \mathcal{Z}}{V} \;\approx\; & -\frac{1}{2} \sum\!\!\!\!\!\!\!\int_P \log(P^2) \;-\; \frac{Z_g^2 g^2}{8} \left(\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} \right)^2 \\ & + \frac{g^4}{16} \left(\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} \right)^2 \sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2)^2} \;+\; \frac{g^4}{48} \sum\!\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2 Q^2 R^2 (P+Q+R)^2}, \end{aligned} \tag{11}$$

where the sum-integral notation is defined in Appendix A. We regularize both ultraviolet and infrared divergences using dimensional regularization in $3-2\epsilon$ spatial dimensions, taking the momentum scale introduced by dimensional regularization to be Λ. In (11) and below, we use the symbol "$\approx$" for an equality that holds only in a strict perturbation expansion in powers of g^2. Such an equality does not properly take into account the screening of infrared divergences at the scale gT, but it can be used for determining short-distance coefficients. The sum-integrals appearing in (11) are given in Appendix A. To the order that is required, renormalization of the coupling constant in the $\overline{\rm MS}$ scheme is accomplished by the substitution

$$Z_g \;=\; 1 \;+\; \frac{3}{4\epsilon} \frac{g^2}{16\pi^2}. \tag{12}$$

After this renormalization, the final result is

$$\frac{T\,\log\mathcal{Z}}{V} \;\approx\; \frac{\pi^2}{9}T^4\Bigg\{\frac{1}{10} \;-\; \frac{1}{8}\frac{g^2(\Lambda)}{16\pi^2} + \frac{1}{8}\left[3\log\frac{\Lambda}{4\pi T}+\frac{31}{15}+\gamma+4\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}\right]\left(\frac{g^2}{16\pi^2}\right)^2\Bigg\},\tag{13}$$

where γ is Euler's constant and $\zeta(z)$ is the Riemann zeta function. The apparent dependence of the right side of (13) on Λ is illusory. The renormalization group equation for the coupling constant,

$$\mu\frac{d}{d\mu}\frac{g^2}{16\pi^2} \;=\; 3\left(\frac{g^2}{16\pi^2}\right)^2 \;+\; O(g^6),\tag{14}$$

implies that the explicit logarithmic dependence on Λ in the g^4 term of (13) is cancelled by the Λ-dependence of the coupling constant in the g^2 term. Thus, up to corrections of order g^6, we can replace Λ on the right side of (13) by an arbitrary renormalization scale μ.

We now consider $\log\mathcal{Z}$ in the effective theory:

$$\log\mathcal{Z} \;=\; -f(\Lambda)\,V \;+\; \log\mathcal{Z}_{\rm eff}.\tag{15}$$

The partition function $\mathcal{Z}_{\rm eff}$ for the effective theory is the functional integral in (7). In the diagrammatic expansion for $\log\mathcal{Z}_{\rm eff}$, the leading terms are given by the diagrams in Fig. 1, plus additional diagrams involving mass insertions as in Fig. 2. To match with the strict expansion in g^2 for the full theory, we should calculate in the effective theory using the perturbation theory defined by the decomposition (10), again using dimensional regularization to regularize both ultraviolet and infrared divergences. This calculation is trivial, since massless loop diagrams with no external legs vanish in dimensional regularization due to a cancellation between ultraviolet poles in ϵ and infrared poles in ϵ. The result is therefore

$$\frac{T\log\mathcal{Z}}{V} \;\approx\; -f\,T.\tag{16}$$

Matching the results (13) and (16), we obtain the coefficient f to order g^4:

$$f \;=\; \frac{\pi^2}{9}T^3\Bigg\{-\frac{1}{10} \;+\; \frac{1}{8}\frac{g^2(\mu)}{16\pi^2} \;-\; \frac{1}{8}\left[3\log\frac{\mu}{4\pi T}+\frac{31}{15}+\gamma+4\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}\right]\left(\frac{g^2}{16\pi^2}\right)^2\Bigg\},\tag{17}$$

where $g^2(\mu)$ is the coupling constant in the $\overline{\mathrm{MS}}$ renormalization scheme at the scale μ. We have used the renormalization group equation (14) to replace Λ in (13) by an arbitrary scale μ associated with the renormalization of the full 4-dimensional theory. Thus, at this order in g^2, the coefficient f does not depend on the ultraviolet cutoff Λ of the effective theory.

IIIb. Mass parameter

In this subsection, we calculate the coefficient $m^2(\Lambda)$ of the $\phi^2/2$ term in the effective lagrangian (6) to next-to-leading order in g^2. The parameter $m(\Lambda)$ can be interpreted as the contribution to the screening mass from short distances of order $1/T$. The actual screening mass m_s is defined by the condition that the propagator for spacelike momentum $K = (k_0 = 0, \mathbf{k})$ has a pole at $\mathbf{k}^2 = -m_s^2$. The physical quantity m_s^2 coincides with $m^2(\Lambda)$ at order g^2, but m_s^2 has corrections of order g^3 which arise from the long-distance scale $1/(gT)$. In contrast, the mass parameter $m^2(\Lambda)$ receives contributions only from the short-distance scale $1/T$, and thus has a perturbative expansion in powers of $g^2(T)$.

One way to determine $m^2(\Lambda)$ is to match the propagator for the zero-frequency mode of the field $\Phi(\mathbf{x}, \tau)$ in the full theory with the propagator for $\phi(\mathbf{x})$ in the effective theory. At leading order in g^2, these operators are related as in (5). This identification is sufficient for determining $m^2(\Lambda)$ to next-to-leading order in g^2. Beyond that order, we must allow for a short-distance coefficient multiplying $\phi(\mathbf{x})$ in (5), and we must also allow for the fact that $\phi(\mathbf{x})$ is only the first term in an operator expansion that contains $\phi^3(\mathbf{x})$ and other higher dimension operators. In general, we must include all operators that are odd under $\phi \to -\phi$, each multiplied by a short-distance coefficient. In order to determine $m^2(\Lambda)$ and all the necessary short-distance coefficients, we would have to match not only the propagator but other 2-point functions as well.

A simpler way to determine $m^2(\Lambda)$ is to match the screening mass in the full theory and in the effective theory. The screening mass gives the location of the pole in the propagator for the zero-frequency mode $\int_0^\beta d\tau \Phi(\mathbf{x}, \tau)$. Denoting the self-energy function for the field

$\Phi(\mathbf{x},\tau)$ at momentum $K=(k_0,\mathbf{k})$ by $\Pi(k_0,\mathbf{k})$, the screening mass m_s is the solution to the equation

$$k^2+\Pi(0,\mathbf{k}) \;=\; 0 \qquad \text{at } k^2=-m_s^2. \tag{18}$$

The location of the pole is independent of field redefinitions. Since the operator expansion that generalizes (5) can be interpreted as a field redefinition, the screening mass m_s also gives the location of the pole in the propagator for the field $\phi(\mathbf{x})$. Denoting the self-energy for $\phi(\mathbf{x})$ by $\Pi_{\rm eff}(k,\Lambda)$, the screening mass m_s must satisfy

$$k^2+m^2(\Lambda)+\Pi_{\rm eff}(k,\Lambda) \;=\; 0 \qquad \text{at } k^2=-m_s^2. \tag{19}$$

By matching the expressions for m_s obtained by solving (18) and (19), we can determine the short-distance parameter $m^2(\Lambda)$.

We will obtain a perturbative expression for the screening mass m_s in the full theory by calculating $\Pi(K)$ to order g^4 using the strict perturbation expansion defined by the decomposition (10). The self-energy is given by the sum of the Feynman diagrams in Fig. 3:

$$\Pi(K) \;\approx\; \frac{Z_g^2 g^2}{2}\, \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} \;-\; \frac{g^4}{4}\, \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2}\, \sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2)^2} \;-\; \frac{g^4}{6}\, \sum\!\!\!\!\!\!\!\int_{PQ} \frac{1}{P^2Q^2(P+Q+K)^2}. \tag{20}$$

We can simplify the equation (18) by expanding $\Pi(0,\mathbf{k})$ as a Taylor expansion around $\mathbf{k}=0$. This can be justified by the fact that the leading order solution to (18) gives a value of k that is of order gT, and $\Pi(K)$ is independent of K at leading order. After setting $K=0$ in the last integral in (20), the solution to (18) to order g^4 is trivial:

$$m_s^2 \;\approx\; \frac{Z_g^2 g^2}{2}\, \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} \;-\; \frac{g^4}{4}\, \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2}\, \sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2)^2} \;-\; \frac{g^4}{6}\, \sum\!\!\!\!\!\!\!\int_{PQ} \frac{1}{P^2Q^2(P+Q)^2}. \tag{21}$$

It should be emphasized that this perturbative expression does not give a physical value for the screening mass, because the sum-integrals are infrared divergent. However, as long as we can compute m_s in the effective theory in a way that makes the same incorrect assumptions about the long distance behavior, we can match the perturbative expressions to determine the short-distance parameter $m^2(\Lambda)$.

In order to match with the expression (21), we have to calculate the screening mass in the effective theory using the strict expansion in g^2 defined by the decomposition (10). The self-energy function $\Pi_{\rm eff}(k,\Lambda)$ in the equation (19) for the screening mass has a diagrammatic expansion including the diagrams in Fig. 3 and the mass-insertion diagrams in Fig. 4. In the full theory, when $\Pi(K)$ is expanded as a Taylor expansion around $K=0$, terms that in dimensional regularization scale like fractional powers of k^2 are automatically set to 0. We should therefore make the same simplifications in the effective theory. But then all the loop diagrams in Fig. 3 and Fig. 4 vanish, since the external momentum $\mathbf{k}$ provides the only mass scale in the integrals. The self-energy function reduces to

$$\Pi_{\rm eff}(k,\Lambda) \;\approx\; \delta m^2, \tag{22}$$

where δm^2 is the mass counterterm that contains the poles in ϵ that are associated with mass renormalization. The solution to the equation (19) for the screening mass is therefore trivial:

$$m_s^2 \;\approx\; m^2(\Lambda) \;+\; \delta m^2. \tag{23}$$

From (23), we see that the screening mass in this unphysical perturbation expansion is just the bare mass. Comparing (21) and (23), we find that $m^2(\Lambda)$ is given by

$$m^2(\Lambda) \;=\; \frac{Z_g^2 g^2}{2} \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} \;-\; \frac{g^4}{4} \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} \sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2)^2} \;-\; \frac{g^4}{6} \sum\!\!\!\!\!\!\!\int_{PQ} \frac{1}{P^2 Q^2 (P+Q)^2} \;-\; \delta m^2, \tag{24}$$

where the sum-integrals are to be evaluated using dimensional regularization of both ultraviolet and infrared divergences. The sum-integrals in (24) are given in Appendix A. After renormalization of the coupling constant g, there remains a pole in ϵ which must be cancelled by the mass counterterm δm^2. The mass counterterm is thereby determined to be

$$\delta m^2 \;=\; \frac{g^4 T^2}{24(16\pi^2)} \frac{1}{\epsilon}. \tag{25}$$

Our final expression for the short-distance mass parameter $m^2(\Lambda)$ is

$$m^2(\Lambda) = \frac{1}{24} g^2(\mu) T^2 \left\{ 1 \;+\; \left[-3 \log\frac{\mu}{4\pi T} + 4 \log\frac{\Lambda}{4\pi T} + 2 - \gamma + 2\frac{\zeta'(-1)}{\zeta(-1)} \right] \frac{g^2}{16\pi^2} \right\}, \tag{26}$$

where $g^2(\mu)$ is the $\overline{\mathrm{MS}}$ coupling constant. We have used the renormalization group equation (14) to change the renormalization scale of the full theory from Λ to μ. The remaining logarithm of Λ in (26) reveals that $m^2(\Lambda)$ depends explicitly on the factorization scale Λ at order g^4. This Λ-dependence is necessary to cancel logarithmic ultraviolet divergences from loop integrals in the effective theory.

IIIc. Coupling constants

For the calculations in this paper, we require the coupling constant λ of the ϕ^4 interaction in the effective theory only to leading order in g^2. At this order, we can simply read λ off from the lagrangian of the full theory. Substituting $\Phi(\tau, \mathbf{x}) \longrightarrow \sqrt{T}\phi(\mathbf{x})$ in (4) and comparing $\int_0^\beta d\tau \mathcal{L}$ with $\mathcal{L}_{\mathrm{eff}}$ in (6), we find that, to leading order in g^2,

$$\lambda \;=\; g^2 T. \tag{27}$$

There is no dependence on the factorization scale Λ at this order. The coupling constant λ could be calculated to higher order in g^2 by matching 4-point correlation functions in the full theory and in the effective theory. Beyond next-to-leading order in g^2, the matching is complicated by the breakdown of the simple relation (5) between $\phi(\mathbf{x})$ and the fundamental field. A more convenient quantity for matching beyond leading order is the on-shell scattering amplitude defined by the residue of the 4-point function at the poles of the propagators of the four external lines. Like the screening mass, this scattering amplitude is invariant under field redefinitions.

The only other coefficient in the effective lagrangian that is known is the coefficient of ϕ^6. It has been computed by Landsman [12], and its value is $15\zeta(3)g^6/(128\pi^4)$. It first contributes to the free energy density at order g^9 and to the square of the screening mass at order g^8.

IV. Calculations in the Effective Theory

In this section, we calculate physical quantities in the effective theory using perturbation theory. We calculate the free energy to order g^5, reproducing a recent result by Parwani and Singh [2], and we obtain a new result for the screening mass to order g^4.

IVa. Free Energy to Order g^5

The free energy density $F(T)$ is defined by

$$\mathcal{Z}(T) \;=\; e^{-\beta\, F(T)\, V}. \tag{28}$$

Comparing with the equivalent expression for the partition function (7), we obtain

$$F(T) = T\; f(\Lambda) - T\; \frac{\log \mathcal{Z}_{\rm eff}}{V}\;, \tag{29}$$

where the partition function for the effective theory is

$$\mathcal{Z}_{\rm eff} \;=\; \int^{(\Lambda)} \mathcal{D}\phi\; \exp\left(- \int d^3x\; \mathcal{L}_{\rm eff} \right). \tag{30}$$

The strict perturbation expansion for $\log \mathcal{Z}_{\rm eff}$ corresponding to the decomposition (10) of $\mathcal{L}_{\rm eff}$ contains infrared divergences. These divergences were not a problem in the matching calculations of Section III, since identical infrared divergences appeared in the strict perturbation expansion for the full theory. However, if we wish to actually calculate the free energy, we must incorporate the physical effects that cut off the infrared divergences into the free part of the lagrangian. The necessary infrared cutoff is provided by the ϕ^2 term in the effective lagrangian. We therefore make the following decomposition of $\mathcal{L}_{\rm eff}$ into free and interacting parts:

$$\begin{aligned}
(\mathcal{L}_{\rm eff})_{\rm free} &\;=\; \frac{1}{2}\,(\boldsymbol{\nabla}\phi)^2 \;+\; \frac{1}{2} m^2(\Lambda)\; \phi^2, \\
(\mathcal{L}_{\rm eff})_{\rm int} &\;=\; \frac{1}{4!} \lambda(\Lambda)\; \phi^4 \;+\; \delta\mathcal{L}.
\end{aligned} \tag{31}$$

From the matching calculations in section III, m^2 is of order g^2T^2 and λ is of order g^2T. Since the only momentum scale in $(\mathcal{L}_{\rm eff})_{\rm free}$ is m, any powers of T in the coefficient of an

operator will be compensated by powers of m. Thus the effective expansion parameter for the ϕ^4 perturbation is λ/m, which is of order g. The next most important perturbation after ϕ^4 is the dimension-4 operator $(\phi\boldsymbol{\nabla}\phi)^2$, for which the effective expansion parameter is of order $g^4 m/T$ or, equivalently, of order g^5. Similarly, the effective expansion parameter for the dimension-3 operator ϕ^6 is g^6. Thus to calculate $\log \mathcal{Z}_{\rm eff}$ to next-to-next-to-leading order in g, we need only consider the ϕ^4 perturbations.

The contributions to $\log \mathcal{Z}_{\rm eff}$ of orders g^3, g^4, and g^5 are given by the sum of the 1-loop, 2-loop, and 3-loop diagrams in Fig. 1 and the first diagram in Fig. 2:

$$\begin{aligned} \frac{\log \mathcal{Z}_{\rm eff}}{V} \;=\; & -\frac{1}{2}\int_p \log(p^2+m_0^2) \;-\; \frac{\lambda}{8}\left(\int_p \frac{1}{p^2+m^2}\right)^2 \\ & + \frac{\lambda^2}{16}\left(\int_p \frac{1}{p^2+m^2}\right)^2 \int_p \frac{1}{(p^2+m^2)^2} \\ & + \frac{\lambda^2}{48}\int_{pqr} \frac{1}{(p^2+m^2)(q^2+m^2)(r^2+m^2)[(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2]}\,. \end{aligned} \qquad (32)$$

The only effect of the diagram with the mass counterterm in Fig. 2 is to replace m^2 in the 1-loop diagram by $m_0 = m^2 + \delta m^2$, where the mass counterterm is

$$\delta m^2 \;=\; \frac{2}{3}\left(\frac{\lambda}{16\pi}\right)^2 \frac{1}{\epsilon}\,. \qquad (33)$$

With the identification $\lambda = g^2 T$, this expression is identical to the expression in (25). When $\int \log(p^2+m_0^2)$ is expanded in powers of λ using (33), the $1/\epsilon$ term cancels against a pole from the last integral in (32), but it also gives rise to finite contributions from the expansion of the integral $\int \log(p^2+m_0^2)$ to order ϵ. The integrals in (32) are given in Appendix B. Adding up the diagrams, we obtain

$$\frac{\log \mathcal{Z}_{\rm eff}}{V} \;=\; \frac{1}{12\pi}m^3(\Lambda) - \frac{1}{8\pi}\frac{\lambda}{16\pi}m^2 \;-\; \frac{1}{12\pi}\left[4\log\frac{\Lambda}{2m} + \frac{9}{2} - 4\log 2\right]\left(\frac{\lambda}{16\pi}\right)^2 m\,. \qquad (34)$$

Our final result for the free energy (29) to order g^5 is the sum of two terms that represent the contributions from the momentum scales T and gT, respectively:

$$F(T) = F_1(T) + F_2(T)\,. \qquad (35)$$

The first term $F_1(T) = fT$ is the contribution to the free energy from single-particle effects involving the short-distance scale $1/T$ only. This contribution can be expressed as a power series in $g^2(2\pi T)$, with the first three terms given by (17):

$$\begin{aligned} F_1(T) &= -\frac{\pi^2}{9} T^4 \left\{ \frac{1}{10} - \frac{1}{8} \frac{g^2(2\pi T)}{16\pi^2} \right. \\ &\left. + \frac{1}{8} \left[\frac{31}{15} - 3\log 2 + \gamma + 4\frac{\zeta'(-1)}{\zeta(-1)} - 2\frac{\zeta'(-3)}{\zeta(-3)} \right] \left(\frac{g^2}{16\pi^2} \right)^2 \right\}. \end{aligned} \quad (36)$$

The choice $\mu = 2\pi T$ for the renormalization scale of the effective theory will be justified in Section V. The second term $F_2(T) = -T \log \mathcal{Z}_{\text{eff}}/V$ in the free energy (35) takes into account collective effects of the particles involving the long-distance scale $1/(gT)$. It can be expressed as a perturbation series in λ and in the other coupling constants of the effective theory. The first three terms are given by (34):

$$F_2(T) = -\frac{1}{12\pi} m^3(\Lambda)\, T \left\{ 1 - \frac{3}{2} \frac{\lambda}{16\pi m} - \left[4\log\frac{\Lambda}{2m} + \frac{9}{2} - 4\log 2 \right] \left(\frac{\lambda}{16\pi m} \right)^2 \right\}, \quad (37)$$

where $\lambda = g^2 T$ and the mass parameter $m(\Lambda)$ is given by (26) with $\mu = 2\pi T$. Note that, after substituting the expression (26) for the mass parameter $m^2(\Lambda)$, the Λ-dependence in (37) cancels to next-to-next-to-leading order in g. Adding the short-distance contribution in (36) and expanding in powers of $g(2\pi T)$, the free energy reduces to

$$\begin{aligned} F(T) &= -\frac{\pi^2}{9} T^4 \left\{ \frac{1}{10} - \frac{1}{8} \left(\frac{g(2\pi T)}{4\pi} \right)^2 + \frac{1}{\sqrt{6}} \left(\frac{g(2\pi T)}{4\pi} \right)^3 \right. \\ &+ \frac{1}{8} \left[-\frac{59}{15} - 3\log 2 + \gamma + 4\frac{\zeta'(-1)}{\zeta(-1)} - 2\frac{\zeta'(-3)}{\zeta(-3)} \right] \left(\frac{g}{4\pi} \right)^4 \\ &\left. + \sqrt{\frac{3}{8}} \left[4\log\frac{g}{4\pi\sqrt{6}} - \frac{5}{2} + 7\log 2 - \gamma + 2\frac{\zeta'(-1)}{\zeta(-1)} \right] \left(\frac{g}{4\pi} \right)^5 \right\}. \end{aligned} \quad (38)$$

The coefficient of g^4 was first calculated by Frenkel, Saa, and Taylor [1], up to an error that was corrected by Arnold and Zhai [5]. The order g^5 term was recently calculated by Parwani and Singh [2]. Our result agrees with theirs after taking into account the difference in the definition of the coupling constant. There is a loss of accuracy when we make a strict expansion in powers of g as in (38). In Section V, we will give a more accurate expression

for the free energy which sums up leading logarithms of g from higher orders of perturbation theory.

IVb. Screening Mass to Order g^4

The screening mass m_s describes the long-distance behavior of the potential produced by the exchange of a particle with spacelike momentum. The potential falls exponentially like $e^{-m_s R}$ at large R. The screening mass is a long distance quantity, so it should be calculable using the effective field theory. In this section, we use the effective field theory to calculate m_s to order g^4.

The screening mass m_s, which gives the location of the pole in the propagator of the effective theory, is the solution to the equation (19). At leading order in λ/m, the solution is simply $m_s = m(\Lambda)$. The self energy function $\Pi_{\rm eff}(k,\Lambda)$ is given to next-to-leading order in λ by the Feynman diagrams in Figs. 3 and 4:

$$\begin{aligned}\Pi_{\rm eff}(k,\Lambda) \;&=\; \frac{\lambda}{2}\int_p \frac{1}{p^2+m^2} \;-\; \frac{\lambda^2}{4}\int_p \frac{1}{p^2+m^2}\int_p \frac{1}{(p^2+m^2)^2} \\ &\quad - \frac{\lambda^2}{6}\int_{pq} \frac{1}{(p^2+m^2)(q^2+m^2)[(\mathbf{p}+\mathbf{q}+\mathbf{k})^2+m^2]} \;+\; \delta m^2 \,. \qquad (39)\end{aligned}$$

The mass counterterm δm^2, which is given in (33), cancels an ultraviolet pole in ϵ in the integral over p and q. This is the only integral in (39) that depends on k. The self-consistent solution to (19) to next-to-leading order in λ/m is obtained by evaluating the integral at the point $k = im$. The resulting expression for the screening mass is

$$\begin{aligned}m_s^2 \;&=\; m^2(\Lambda) \;+\; \frac{\lambda}{2}\int_p \frac{1}{p^2+m^2} \;-\; \frac{\lambda^2}{4}\int_p \frac{1}{p^2+m^2}\int_p \frac{1}{(p^2+m^2)^2} \\ &\quad - \frac{\lambda^2}{6}\int_{pq} \frac{1}{(p^2+m^2)(q^2+m^2)[(\mathbf{p}+\mathbf{q}+\mathbf{k})^2+m^2]}\Bigg|_{k=im} \;+\; \delta m^2 \,. \qquad (40)\end{aligned}$$

Note that the screening mass is not identical to the value of the inverse propagator at 0 momentum, which would be given by (40) with the last integral evaluated at $k = 0$. Unlike the screening mass, the mass defined by the inverse propagator at $k = 0$ is not invariant under field redefinitions.

The integrals in (40) are given in Appendix B. To next-to-next-to-leading order in λ/m, the screening mass is

$$m_s^2 \;=\; m^2(\Lambda)\left\{1 \;-\; 2\,\frac{\lambda}{16\pi m} \;-\; \frac{2}{3}\left[4\log\frac{\Lambda}{2m}+3-8\log 2\right]\left(\frac{\lambda}{16\pi m}\right)^2\right\}. \tag{41}$$

Note that, when we substitute (26) for $m^2(\Lambda)$, the Λ-dependence cancels to next-to-next-to-leading order in g. This verifies that the screening mass is independent of the arbitrary factorization scale Λ to this order. Setting $\mu = 2\pi T$ in (26), substituting it into (41), and expanding in powers of $g(2\pi T)$, we obtain

$$m_s^2 \;=\; \frac{1}{24}\,g^2(2\pi T)\,T^2\left\{1-\sqrt{6}\,\frac{g}{4\pi} \;+\; \left[4\log\frac{g}{4\pi\sqrt{6}}-1+11\log 2-\gamma+2\frac{\zeta'(-1)}{\zeta(-1)}\right]\left(\frac{g}{4\pi}\right)^2\right\}. \tag{42}$$

The term of order g^3 in (42) was first calculated by Dolan and Jackiw [16]. The correction of order g^4 in the expression for the screening mass is a new result. The screening mass differs at order g^4 from the quasiparticle mass, which was calculated to order g^4 by Parwani [15]. The quasiparticle mass is defined by the pole in the energy for the propagator at zero 3-momentum. It gives the energy of the single-particle excitations in the plasma, while the screening mass gives the range of the force mediated by the exchange of a particle in the plasma. There is a loss of accuracy in making a strict expansion of the screening mass in powers of g as in (42). In Section V, we will give a more accurate expression for the screening mass which resums leading logarithms of g from higher orders of perturbation theory.

V. Summation of Leading Logarithms

The coefficients in the effective lagrangian (6) depend on not only on the coupling constant g^2 and the temperature T, but also on two arbitrary momentum scales: the renormalization scale μ of the full theory and the ultraviolet cutoff Λ of the effective theory. In this section, we exploit these arbitrary scales to sum up the leading logarithms from higher orders in perturbation theory. We first discuss the choice of the renormalization scale μ. We then

present the evolution equations that describe the dependence of the short-distance coefficients on the scale Λ. We then show how the solutions of the evolution equations can be used to sum up leading logarithms of $T/(gT)$ in physical quantities.

Va. Renormalization scale

The short-distance coefficients in the effective lagrangian would be independent of the arbitrary renormalization scale μ of the original theory if they were calculated to all orders in g^2. A dependence on μ appears however when a coefficient is calculated only to a finite order in g^2. At leading nontrivial order in g^2, the scale μ appears to be completely arbitrary. The resulting ambiguity can be decreased by a next-to-leading order calculation, because an inappropriate choice of μ will result in unnecessarily large perturbative corrections. The short-distance coefficients f and $m^2(\Lambda)$ were both calculated to order g^4 in Section III, and we can use those results to discuss the appropriate scale for μ in the g^2 terms.

The transcendental constants appearing in the expressions (17) and (26) have the numerical values

$$\gamma \;=\; 0.57722, \qquad \frac{\zeta'(-1)}{\zeta(-1)} \;=\; 1.98505, \qquad \frac{\zeta'(-3)}{\zeta(-3)} \;=\; 0.64543. \tag{43}$$

The short-distance coefficients then reduce to

$$f \;=\; -\frac{\pi^2}{90}T^3\left\{1 \;-\; \frac{5}{4}\frac{g^2(\mu)}{16\pi^2}\left[1 \;+\; \left(-7.2 - 3\log\frac{\mu}{2\pi T}\right)\frac{g^2}{16\pi^2}\right]\right\}, \tag{44}$$

$$m^2(\Lambda) \;=\; \frac{1}{24}g^2(\mu)T^2\left[1 \;+\; \left(4.7 - 3\log\frac{\mu}{2\pi T} + 4\log\frac{\Lambda}{2\pi T}\right)\frac{g^2}{16\pi^2}\right], \tag{45}$$

where the coupling constant is the $\overline{\rm MS}$ constant at the scale μ. This coupling constant is defined so that it is the appropriate coupling constant for particles whose Euclidean invariant mass-squared $k_0^2+k^2$ is approximately μ^2. The parameters (44) and (45) are coefficients in an effective lagrangian obtained by integrating out modes with nonzero Matsubara frequencies $k_0 = 2n\pi T$, $n = 1, 2, 3, \ldots$. Since these modes have invariant masses that are equal to or greater than $2\pi T$, we expect $\mu = 2\pi T$ to be an appropriate choice for the renormalization scale. Choosing $\mu = \Lambda = 2\pi T$, the coefficients of $g^2/(16\pi^2)$ in square brackets in (44) and

in (45) are -7.2 and 4.7, respectively, which are reasonably small. The more naive choice $\mu = \Lambda = T$ gives even smaller coefficients -1.7 and 2.9. We prefer the physically motivated choice, and we therefore set the renormalization scale to $\mu = 2\pi T$ throughout the remainder of this paper.

Vb. Factorization scale

The matching calculations described in Section III give the short-distance coefficients in the effective lagrangian $\mathcal{L}_{\rm eff}$ as perturbation series in $g^2(2\pi T)$ with coefficients that are polynomials in $\log(2\pi T/\Lambda)$. To avoid unnecessarily large coefficients in these perturbative expansions, we must choose Λ of order $2\pi T$. Once the short-distance coefficients are known, physical quantities can be calculated in the effective theory using a perturbation expansion in λ/m and in other dimensionless parameters obtained by dividing the short-distance parameters by appropriate powers of m. The coefficients in the resulting perturbation expansions for physical quantities contain logarithms of Λ/m. To avoid unnecessarily large coefficients in these perturbation expansions, it is necessary to carry out the perturbative calculations in the effective theory using short-distance parameters that are evaluated at a scale Λ of order m. The parameters calculated at the original scale $\Lambda = 2\pi T$ must therefore be evolved down to the scale $\Lambda = m$ before they can be used in these perturbative calculations. The Λ-dependence of these parameters is described by "renormalization group equations" or "evolution equations".

The effective lagrangian (6) can be expressed as a sum over over all local operators that respect the symmetries of the theory:

$$f(\Lambda) \;+\; \mathcal{L}_{\rm eff} \;=\; \sum_n C_n(\Lambda)\, \mathcal{O}_n, \tag{46}$$

where we have included the unit operator as one of the operators $\mathcal{O}_n$. The coefficients C_n are the generalized coupling constants of the effective theory. Because of ultraviolet divergences, the effective theory must be regularized with an ultraviolet cutoff Λ. The ultraviolet

divergences in the effective theory include power ultraviolet divergences proportional to Λ^p, $p = 1, 2, \ldots$, and logarithmic divergences proportional to $\log(\Lambda/m)$. As discussed in Section II, the power divergences are artifacts of the regularization scheme and have no physical content. If they are not removed as part of the regularization procedure, they must be cancelled by power divergences in the coupling constants C_n. In contrast, the logarithmic ultraviolet divergences are directly related to logarithms of T in the full theory, and therefore represent real physical effects. This difference justifies treating power ultraviolet divergences and the logarithmic ultraviolet divergences differently. It is convenient to use a regularization procedure for the effective theory in which power ultraviolet divergences are subtracted and logarithmically ultraviolet divergent integrals are cut off at the scale Λ. These logarithmic divergences are then the only ones that must be cancelled by the Λ-dependence of the coupling constants C_n. The dimensions of a coupling constant can then only be taken up by powers of the temperature T. The coupling constant C_n must be proportional to T^{3-d_n}, where d_n is the scaling dimension of the corresponding operator $\mathcal{O}_n$. The dimensionless factor multipling T^{3-d_n} in the coupling constant C_n can be computed as a perturbation series in $g^2(T)$, with coefficients that are polynomials in $\log(T/\Lambda)$. The dependence on Λ is governed by a "renormalization group equation" or "evolution equation" of the form

$$\Lambda \frac{d}{d\Lambda} C_n(\Lambda) \ = \ \beta_n(C), \tag{47}$$

where the beta function β_n has a power series expansion in the coupling constants C_m. These equations follow from the condition that physical quantities must be independent of the arbitrary scale Λ.

Since C_n is proportional to T^{3-d_n}, every term in the expansion of its beta function must be proportional to T^{3-d_n}. In particular, a term like $C_{m_1} C_{m_2} \ldots C_{m_k}$ can appear only if the dimensions d_{m_i} of the corresponding operators $\mathcal{O}_{m_i}$ satisfy

$$\sum_{i=1}^{k} (3 - d_{m_i}) \ = \ 3 - d_n. \tag{48}$$

This condition is very restrictive, particularly if the effective field theory is truncated to those

terms that are given explicitly in (6). It implies that the beta function for the coefficient f of the unit operator can only have two terms proportional to λ^3 and $m^2\lambda$. The beta function for m^2 must be proportional to λ^2 and the beta function for λ must vanish to all orders in λ. These restrictions reflect the super-renormalizability of this truncated effective theory, which implies that there are only a finite number of independent ultraviolet-divergent subdiagrams.

The evolution equations for f and m^2 are calculated in Appendix C. They follow from the condition that the free energy and the screening mass must be independent of Λ. The evolution equations are

$$\Lambda \frac{d}{d\Lambda} f = -\frac{\pi}{12}\left(\frac{\lambda}{16\pi}\right)^3, \tag{49}$$

$$\Lambda \frac{d}{d\Lambda} m^2 = \frac{8}{3}\left(\frac{\lambda}{16\pi}\right)^2, \tag{50}$$

$$\Lambda \frac{d}{d\Lambda} \lambda = 0. \tag{51}$$

The allowed term $m^2\lambda$ does not appear in the beta function for f. Using $\lambda = g^2 T$, we see that the beta function for m^2 in (50) is consistent with the explicit calculation to order g^4 in (26). The evolution equations (49), (50), and (51) are correct to all orders in λ, but they receive corrections involving the coefficients of higher dimension operators like ϕ^6. The coefficient of ϕ^6 in the effective lagrangian is of order g^6. It gives corrections to the right sides of the evolution equations (49), (50), and (51) that are of order $g^{12}\lambda^3$, $g^6\lambda^2$, and $g^6\lambda$, respectively.

To the accuracy given in (49), (50), and (51), the solutions to the evolution equations are trivial. The coupling constant λ does not evolve with Λ, and the solutions for f and m^2 are simply linear in $\log(\Lambda)$:

$$f(\Lambda) = f(2\pi T) - \frac{\pi}{12}\left(\frac{\lambda}{16\pi}\right)^3 \log(\Lambda/2\pi T), \tag{52}$$

$$m^2(\Lambda) = m^2(2\pi T) + \frac{8}{3}\left(\frac{\lambda}{16\pi}\right)^2 \log(\Lambda/2\pi T). \tag{53}$$

Note that, since the perturbative expression (26) is linear in $\log(\Lambda)$, it already satisfies the evolution equation (50).

Vc. Resumming Logarithms of g

The evolution equations for the short-distance coefficients can be used to sum up leading logarithms of $T/(gT)$ in physical quantities, such as the free energy and the screening mass. We must first choose a value for the scale Λ which will avoid unnecessarily large coefficients in the perturbation expansions of the effective theory. A reasonable choice is the screening mass m_s, since this is the minimum invariant mass for a particle in the 3-dimensional Euclidean theory. At leading order in g, the screening mass is simply $m_s = gT/\sqrt{24}$, so we choose $\Lambda = gT/\sqrt{24}$.

An expansion for the screening mass in powers of g is given in (42). This expression is only accurate up to corrections of order $g^5 \log g$. A more accurate expression for the screening mass can be obtained by not expanding out the mass parameter in (41):

$$m_s^2 \;=\; m^2 \left\{ 1 \;-\; 2\,\frac{g^2 T}{16\pi m} \;-\; \frac{2}{3}\left[3 - 12\log 2\right]\left(\frac{g^2 T}{16\pi m}\right)^2 \right\}, \tag{54}$$

where $g = g(2\pi T)$ is the coupling constant at the scale $2\pi T$ and m^2 is the mass parameter in (26) with $\mu = 2\pi T$ and $\Lambda = gT/\sqrt{24}$:

$$m^2 \;=\; \frac{1}{24} g^2\, T^2 \left\{ 1 \;+\; \left[4\log\frac{g}{4\pi\sqrt{6}} + 2 - \log 2 - \gamma + 2\frac{\zeta'(-1)}{\zeta(-1)} \right] \frac{g^2}{16\pi^2} \right\}. \tag{55}$$

The expression (54) is accurate up to corrections of order g^5. It also sums up all the "leading logarithms" of order $g^{2n+3} \log^n g$, $n = 2, 3, \ldots$. These terms are generated by expanding out the term proportional to $g^2 T m$ in (54) using (55).

An expansion for the free energy in powers of g is given in (38). This expression is accurate up to corrections of order $g^6 \log g$. A more accurate result can be obtained by using the solution to the renormalization group equation (49) for f in the short-distance part and by not expanding out the mass parameter m in the long-distance part (37):

$$\begin{aligned} F(T) \;&=\; F_1(T) \;-\; \frac{\pi}{12}\left(\frac{g^2}{16\pi}\right)^3 T^4 \log\frac{g}{4\pi\sqrt{6}} \\ &\quad -\; \frac{1}{12\pi}\, m^3\, T \left\{ 1 \;-\; \frac{3}{2}\frac{g^2 T}{16\pi m} \;-\; \left[\frac{9}{2} - 8\log 2\right]\left(\frac{g^2 T}{16\pi m}\right)^2 \right\}, \end{aligned} \tag{56}$$

where $g = g(2\pi T)$, and m is given by (55). The first two terms on the right side of (56) are the short distance contribution $f(\Lambda)T$, with the factorization scale evaluated at $\Lambda = gT/\sqrt{24}$. Its expansion to order g^4 is given by $F_1(T)$ in (36), and the term of order $g^6 \log g$ comes from the solution (52) to the renormalization group equation for f. There is also another contribution of order $g^6 \log g$ that comes from expanding out the $g^2 m^2 T^2$ term in (56) using (55). Having included both of these $g^6 \log g$ terms, the expression (56) for the free energy is accurate up to corrections of order g^6. It also sums up all the "leading logarithms" of the form $g^{2n+3} \log^n g$, $n = 2, 3, \ldots$, which are obtained by expanding out the $m^3 T$ term in (56).

VI. Conclusions

We have developed an effective-field-theory approach for calculating the thermodynamic properties of a field theory in the high temperature limit. The effective field theory is the 3-dimensional theory obtained by dimensional reduction to the bosonic zero-frequency modes. The short-distance coefficients in the effective lagrangian are computed by straightforward perturbative calculations in the full theory without any resummation. Thermodynamic quantities are then calculated using a perturbation expansion in the effective theory which incorporates the effects of screening. In each of these two steps, the calculations involve only a single mass scale, which greatly simplifies the sums and integrals that need to be evaluated. The short-distance coefficients satisfy renormalization group equations which can be used to improve the perturbation expansion by summing up leading logarithms of $T/(gT)$. The power of this method was demonstrated by carrying out two calculations in massless Φ^4 theory beyond the highest orders that were previously available. The free energy was calculated to order $g^6 \log g$ and the screening mass was calculated to order $g^5 \log g$.

The method that we have used to calculate the free energy of a scalar field theory can also be applied to nonabelian gauge theories, such as QCD. The free energy for QCD has been calculated with an error of order $g^5 \log g$ using more conventional resummation methods [5]. Using our effective field theory approach, it should be straightforward to decrease the error

to order g^6. The effective theory obtained by integrating out the scale T is a 3-dimensional gauge theory with a scalar field in the adjoint representation. There are two short distance coefficients that must be calculated beyond leading order in g^2 in order to compute the free energy to order g^5. The coefficient f of the unit operator is required to next-to-next-to-leading order in g^2, but this is already known [8]. The electric mass parameter $m_{\rm el}^2$ must be calculated to next-to-leading order in g^2. The error can be decreased further by calculating the renormalization group equations for f amd $m_{\rm el}^2$. Their solutions can be used to sum up terms of the form $g^6 \log^2 g$ and $g^6 \log g$, thereby reducing the error to order g^6. This is the maximal accuracy that can be achieved using purely diagrammatic methods. At order g^6, there is a contribution to the free energy from the momentum scale $g^2 T$ that can only be calculated using lattice simulations of 3-dimensional QCD [8].

A similar effective field theory approach was recently developed by Farakos, Kajantie, Rummukainen, and Shaposhnikov to study the electroweak phase transition [17]. They integrated out the scale T to get a 3-dimensional effective theory, and exploited the renormalization group equations of the effective theory to take into account logarithms of $T/(gT)$, just as we have done in this paper. They also integrating out the scale gT to obtain a second effective field theory that must be treated numerically. This second step is necessary in gauge theories since there is no screening of magnetostatic forces at the scale gT. This strategy has also been used to determine the asymptotic behavior of the correlator of Polyakov loops [9] and to solve the problem of calculating the magnetostatic contribution to the free energy of a nonabelian gauge theory [8].

An effective field theory approach has also been applied recently to the massless Φ^4 theory by Marini and Burgess [18]. They used a momentum cutoff as a regulator and their short-distance coefficients therefore contain power ultraviolet divergences that serve simply to cancel the power ultraviolet divergences from loop integrals. These power divergences greatly complicate the renormalization group equations for the short-distance coefficients. As we have emphasized, the power divergences are artifacts of the regulator and might as

well be subtracted as part of the regularization scheme. This not only greatly streamlines explicit calculations, but it also makes the conceptual framework more transparent.

Effective field theories obtained by dimensional reduction have provided great insight into the qualitative behavior of field theories in the high temperature limit. In this paper, we have shown how they can also be used as a practical tool for explicit calculations. By using effective field theory to separate the important momentum scales T and gT (and g^2T if necessary), perturbative calculations can be organized into steps that involve only a single momentum scale at a time. By exploiting the renormalization group structure of the effective theory to sum up leading logarithms of g, potentially large coefficients in the perturbative expansion can be brought under control. We have exhibited the power of our effective-field-theory method by carrying out pioneering calculations in massless Φ^4 theory. This method has many other exciting applications, especially in unravelling the complexities of nonabelian gauge theories at high temperature.

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A Sum-integrals in the Full Theory

In the imaginary-time formalism for thermal field theory, a boson has Euclidean 4-momentum $P = (p_0, \mathbf{p})$, with $P^2 = p_0^2 + \mathbf{p}^2$. The Euclidean energy p_0 has discrete values: $p_0 = 2\pi n T$, where n is an integer. Loop diagrams involve sums over p_0 and integrals over $\mathbf{p}$. It is convenient to introduce a concise notation for these sums and integrals. If dimensional

regularization is used to regularize ultraviolet or infrared divergences, the definition is

$$\sum\!\!\!\!\!\!\!\!\int_P \;\equiv\; \left(\frac{e^\gamma \mu^2}{4\pi}\right)^\epsilon \, T \sum_{p_0} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}\,, \tag{A.1}$$

where $3-2\epsilon$ is the dimension of space and μ is an arbitrary momentum scale. The factor $(e^\gamma/4\pi)^\epsilon$ is introduced so that, after minimal subtraction of the poles in ϵ due to ultraviolet divergences, μ coincides with the renormalization scale of the $\overline{\mathrm{MS}}$ renormalization scheme.

The sum-integrals required to calculate the coefficient $f(\Lambda)$ to next-to-leading order in g^2 can be found in Ref. [5]. We reproduce them here for convenience:

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\!\int_P \log P^2 \;&=\; -\frac{\pi^2 T^4}{45}\left[1+O(\epsilon)\right]\,, &\text{(A.2)}\\
\sum\!\!\!\!\!\!\!\!\int_P \frac{1}{P^2} \;&=\; \frac{T^2}{12}\left[1+\epsilon\left(2\log\frac{\mu}{4\pi T}+2+2\frac{\zeta'(-1)}{\zeta(-1)}\right)+O(\epsilon^2)\right]\,, &\text{(A.3)}\\
\sum\!\!\!\!\!\!\!\!\int_P \frac{1}{(P^2)^2} \;&=\; \frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon}+2\log\frac{\mu}{4\pi T}+2\gamma+O(\epsilon)\right]\,, &\text{(A.4)}\\
\sum\!\!\!\!\!\!\!\!\int_{PQ} \frac{1}{P^2Q^2(P+Q)^2} \;&=\; 0\,, &\text{(A.5)}\\
\sum\!\!\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2Q^2R^2(P+Q+R)^2} \;&=\; \frac{T^4}{24(4\pi)^2}\left[\frac{1}{\epsilon}+6\log\frac{\mu}{4\pi T}\right.\\
&\qquad \left.+\frac{91}{15}+8\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}+O(\epsilon)\right]\,, &\text{(A.6)}
\end{aligned}$$

where γ is Euler's constant and $\zeta(z)$ is Riemann's zeta function.

B Integrals in the Effective Theory

The effective theory for the scale g^2T is an Euclidean field theory in 3 space dimensions. Loop diagrams involve integrals over 3-momenta. It's convenient to introduce the notation $\int_p$ for these integrals. If dimensional regularization in $3-2\epsilon$ dimensions is used to regularize ultraviolet divergences, we use the integration measure

$$\int_p \;\equiv\; \left(\frac{e^\gamma \mu^2}{4\pi}\right)^\epsilon \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}\,. \tag{B.1}$$

If renormalization is accomplished by the minimal substraction of poles in ϵ, then μ is the renormalization scale in the $\overline{\mathrm{MS}}$ scheme.

The integrals that are required to calculate the free energy to order g^5 and the screening mass to order g^4 are

$$\int_p \log(p^2+m^2) = -\frac{m^3}{6\pi}\left[1+\epsilon\left(2\log\frac{\mu}{2m}+\frac{8}{3}\right)+O(\epsilon^2)\right], \tag{B.2}$$

$$\int_p \frac{1}{p^2+m^2} = -\frac{m}{4\pi}\left[1+\epsilon\left(2\log\frac{\mu}{2m}+2\right)+O(\epsilon^2)\right], \tag{B.3}$$

$$\int_p \frac{1}{(p^2+m^2)^2} = \frac{1}{8\pi m}\left[1+\epsilon\left(2\log\frac{\mu}{2m}\right)+O(\epsilon^2)\right], \tag{B.4}$$

$$\int_{pq}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2} = \frac{1}{(8\pi)^2}\left[\frac{1}{\epsilon}+4\log\frac{\mu}{2m}+2+4\log 2-4\log 3+O(\epsilon)\right], \tag{B.5}$$

$$\int_{pq}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{k})^2+m^2}\bigg|_{k=im} = \frac{1}{(8\pi)^2}\left[\frac{1}{\epsilon}+4\log\frac{\mu}{2m}+6-8\log 2+O(\epsilon)\right], \tag{B.6}$$

$$\int_{pqr}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} = -\frac{m}{(4\pi)^3}\left[\frac{1}{\epsilon}+6\log\frac{\mu}{2m}+8-4\log 2+O(\epsilon)\right], \tag{B.7}$$

The integral (B.2) is standard. The remaining integrals are most easily evaluated by going to coordinate space. The Fourier transform of the propagator $1/(k^2+m^2)$ in $3-2\epsilon$ dimensions defines a potential $V(R)$:

$$V(R) \equiv \int_k e^{i\mathbf{k}\cdot\mathbf{R}}\,\frac{1}{k^2+m^2}\,. \tag{B.8}$$

It can be expressed in terms of the modified Bessel function $K_\nu(z)$:

$$V(R) = \left(\frac{e^\gamma\mu^2}{4\pi}\right)^\epsilon \frac{1}{(2\pi)^{3/2-\epsilon}}\left(\frac{m}{R}\right)^{1/2-\epsilon} K_{1/2-\epsilon}(mR)\,. \tag{B.9}$$

At $\epsilon=0$, this reduces to the familiar Coulomb potential

$$V_0(R) = \frac{e^{-mR}}{4\pi R}\,. \tag{B.10}$$

For small R, the potential $V(R)$ can be expressed as the sum of two Laurent expansions in R^2, a singular one beginning with an $R^{-1+2\epsilon}$ term and a regular one begining with an R^0

term:

$$V(R) = \left(\frac{e^\gamma \mu^2}{4}\right)^\epsilon \frac{\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(\frac{1}{2})} \frac{1}{4\pi} R^{-1+2\epsilon} \left[1 + \frac{m^2 R^2}{2(1+2\epsilon)} + O(m^4 R^4)\right]$$
$$- (e^\gamma \mu^2)^\epsilon \frac{\Gamma(-\frac{1}{2}+\epsilon)}{\Gamma(-\frac{1}{2})} \frac{1}{4\pi} m^{1-2\epsilon} \left[1 + \frac{m^2 R^2}{2(3-2\epsilon)} + O(m^4 R^4)\right]. \qquad (B.11)$$

The integrals (B.3) and (B.4) are related to the potential at $R = 0$. By the rules of dimensional regularization, they can be evaluated by taking ϵ large enough that the $R^{-1+2\epsilon}$ term vanishes as $R \to 0$, and then analytically continuing back to small values of ϵ. The two integrals are

$$\int_p \frac{1}{p^2+m^2} \equiv V(0) = -(e^\gamma \mu^2)^\epsilon \frac{\Gamma(-\frac{1}{2}+\epsilon)}{\Gamma(-\frac{1}{2})} \frac{1}{4\pi} m^{1-2\epsilon}, \qquad (B.12)$$
$$\int_p \frac{1}{(p^2+m^2)^2} \equiv -\frac{1}{2m}\frac{d}{dm} V(R)\bigg|_{R=0} = (e^\gamma \mu^2)^\epsilon \frac{\Gamma(\frac{1}{2}+\epsilon)}{\Gamma(\frac{1}{2})} \frac{1}{8\pi} m^{-1-2\epsilon}. \qquad (B.13)$$

The integrals (B.6) and (B.7) require a little more effort. The integral (B.7) can be written

$$\int_{pqr} \frac{1}{p^2+m^2} \frac{1}{q^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} = \int_R V^4(R), \qquad (B.14)$$

where $\int_R$ is defined by

$$\int_R \equiv \left(\frac{e^\gamma \mu^2}{4\pi}\right)^{-\epsilon} \int d^{3-2\epsilon} R\,. \qquad (B.15)$$

From the $R \to 0$ region of the integral (B.14), there is a linear ultraviolet divergence, which is removed by dimensional regularization, and a logarithmic divergence, which appears as a pole in ϵ. We evaluate the integral by splitting the radial integration into two regions, $0 < R < r$ and $r < R < \infty$. Since the ultraviolet divergences come only from the region $R \to 0$, we can set $\epsilon = 0$ in the region $r < R < \infty$. Thus the integral can be written

$$\int_R V^4(R) = \left(\frac{e^\gamma \mu^2}{4}\right)^{-\epsilon} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}-\epsilon)}\, 4\pi \int_0^r dR\, R^{2-2\epsilon}\, V^4(R) + 4\pi \int_r^\infty dR\, R^2\, V_0^4(R)\,. \qquad (B.16)$$

By choosing $r \ll 1/m$, we can evaluate the first integral on the right side of (B.16) by using the small-R expansion (B.11) for $V(R)$. Dropping all terms that vanish as $r \to 0$, we get

$$\left(\frac{e^\gamma \mu^2}{4}\right)^{-\epsilon} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}-\epsilon)}\, 4\pi \int_0^r dR\, R^{2-2\epsilon}\, V^4(R)$$
$$= \frac{1}{(4\pi)^3}\left[-\frac{1}{r} - m\left(\frac{1}{\epsilon} + 2\log\frac{r^2\mu^3}{2m} + 4\gamma + 4\right)\right] + O(\epsilon)\,. \qquad (B.17)$$

The integral over the region $r < R < \infty$ is easily evaluated using integration by parts. Dropping all terms that vanish as $r \to 0$, we get

$$4\pi \int_r^\infty dR\, R^2\, V^4(R) = \frac{1}{(4\pi)^3} \left[\frac{1}{r} + 4m \left(\log 4mr + \gamma - 1 \right) \right] . \tag{B.18}$$

Note that the $1/r$ and $\log r$ terms cancel between (B.17) and (B.18). Inserting (B.17) and (B.18) into (B.16), we obtain (B.7).

The integral (B.6) can be evaluated in a similar way to (B.7). It can be written

$$\int_{pq} \frac{1}{p^2+m^2} \frac{1}{q^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{k})^2+m^2} = \int_R e^{i\mathbf{k}\cdot\mathbf{R}}\, V^3(R) . \tag{B.19}$$

Again we split the radial integration into two regions and set $\epsilon = 0$ in the region $r < R < \infty$. After evaluating the angular integrals, we obtain

$$\begin{aligned} \int_R e^{i\mathbf{k}\cdot\mathbf{R}}\, V^3(R) \quad &= \quad \left(\frac{e^\gamma \mu^2}{2k} \right)^{-\epsilon} \frac{(2\pi)^{3/2}}{\sqrt{k}} \int_0^r dR\; R^{3/2-\epsilon} J_{1/2-\epsilon}(kR)\; V^3(R) \\ &\qquad + \frac{4\pi}{k} \int_r^\infty dR\; R \sin(kR)\; V_0^3(R) , \end{aligned} \tag{B.20}$$

where $J_\nu(z)$ is an ordinary Bessel function. We evaluate the first integral using the small-R expansion (B.11) for $V(R)$ and the small-R expansion for the Bessel function

$$J_{1/2-\epsilon}(kR) = \frac{1}{\Gamma(\frac{3}{2}-\epsilon)} \left(\frac{1}{2} kR \right)^{1/2-\epsilon} \left[1 + O(k^2 R^2) \right] . \tag{B.21}$$

Dropping terms that vanish as $r \to 0$, the first integral in (B.20) is

$$\begin{aligned} &\left(\frac{e^\gamma \mu^2}{2k} \right)^{-\epsilon} \frac{(2\pi)^{3/2}}{\sqrt{k}} \int_0^r dR\; R^{3/2-\epsilon} J_{1/2-\epsilon}(kR)\; V^3(R) \\ &\qquad = \quad \frac{1}{(8\pi)^2} \left(\frac{1}{\epsilon} + 4 \log \mu r + 2 + 4\gamma \right) \; + \; O(\epsilon) . \end{aligned} \tag{B.22}$$

This integral is independent of k. In the second integral in (B.20), we have to set $k = im$:

$$\frac{4\pi}{k} \int_r^\infty dR\; R \sin(kR)\; V_0^3(R) \bigg|_{k=im} \quad = \quad \frac{1}{(4\pi)^2} \left(-\log 2mr + 1 - 2\log 2 - \gamma \right) . \tag{B.23}$$

Adding (B.22) and (B.23), the logarithms of r cancel and we obtain (B.6).

C Evolution Equations for Coefficients

In this appendix, we calculate the evolution equations for the short-distance coefficients f and m^2 in the effective lagrangian. These equations follow from the condition that physical quantities must be independent of the arbitrary renormalization scale Λ of the effective theory. The Λ-dependence of the parameters in the effective lagrangian must cancel the Λ-dependence from loop integrals. If power ultraviolet divergences are subtracted as part of the regularization scheme, then the Λ-dependence comes only from logarithmically-divergent loop integrals. One-loop diagrams in a 3-dimensional field theory never give logarithmic ultraviolet divergences. After averaging over angles, such an integral has the behavior $\int d^3p/(p^2)^n$ at large p. This gives a power divergence for $n = 1$ or less and is convergent for $n = 2$ or greater. Thus logarithmic divergences only arise in diagrams with 2 or more loops. If the number of loops is odd, logarithmic divergences only arise from subdiagrams with an even number of loops. Thus the evolution equations are completely determined by diagrams with an even number of loops.

The renormalization group equations for m^2 can be determined from the condition that the screening mass m_s is independent of Λ. Since only the ultraviolet region of loop integrals is relevant for determining the evolution equations, we can use the expression for the screening mass that is obtained from the strict perturbation expansion in g^2 defined by the decomposition (10). The expression for the screening mass to order g^4 is

$$m_s^2 \;\approx\; m^2 \;+\; \frac{\lambda}{2}\int_p \frac{1}{p^2} \;-\; \frac{\lambda^2}{4}\int_p \frac{1}{p^2}\int_p \frac{1}{(p^2)^2} \;-\; \frac{\lambda^2}{6}\int_{pq} \frac{1}{p^2 q^2 (\mathbf{p}+\mathbf{q})^2} \;-\; \frac{\lambda m^2}{2}\int_p \frac{1}{(p^2)^2} \;+\; \delta m^2. \tag{C.1}$$

The only logarithmic divergence comes from the 2-loop integral over p and q, which comes from the last diagram in Fig. 3. Thus the condition that m_s^2 is independent of Λ reduces to

$$\Lambda\frac{d}{d\Lambda}m^2 \;=\; \frac{\lambda^2}{6}\,\Lambda\frac{d}{d\Lambda}\int_{pq} \frac{1}{p^2 q^2 (\mathbf{p}+\mathbf{q})^2}. \tag{C.2}$$

The ultraviolet divergence in the integral on the right side of (C.2) is the same as in (B.5). The derivative is with respect to the scale Λ associated with the ultraviolet cutoff, and

must be taken with the infrared cutoff fixed. The integral in (C.2) vanishes in dimensional regularization only if we use the same regularization scale μ for ultraviolet and infrared divergences. If we use a different scale Λ for the regularization of ultraviolet divergences, the integral is

$$\int_{pq} \frac{1}{p^2 q^2 (\mathbf{p}+\mathbf{q})^2} \;=\; \frac{1}{(8\pi)^2}\left[\left(\frac{1}{\epsilon}\right)_{UV} - \left(\frac{1}{\epsilon}\right)_{IR} + 4\log\frac{\Lambda}{\mu}\right]. \tag{C.3}$$

The subscripts UV and IR indicate whether the pole in ϵ is of ultraviolet or infrared origin. Inserting (C.3) into (C.2), we obtain the evolution equation

$$\Lambda\frac{d}{d\Lambda}m^2 \;=\; \frac{8}{3}\left(\frac{\lambda}{16\pi}\right)^2. \tag{C.4}$$

This result is accurate to all orders in λ and to leading order in the coefficients of higher dimension operators.

The evolution equation for f can be determined from the condition that the free energy, or equivalently, the logarithm of the partition function given in (15), is independent of Λ. We need only consider diagrams for $\log\mathcal{Z}_{\rm eff}$ which have an even number of loops, and they can be calculated using the strict perturbation expansion in g^2 defined by the decomposition (10). The two-loop diagram for $\log\mathcal{Z}_{\rm eff}$ in Fig. 1 has no logarithmic ultraviolet divergence. The four-loop diagrams that have logarithmic ultraviolet divergences are shown in Fig. 5. The first diagram in Fig. 5 has two-loop subdiagrams that are logarithmically divergent. The resulting Λ-dependence is cancelled by the Λ-dependence of m^2 in the coefficient of the second diagram of Fig. 2. The second diagram in Fig. 5 has no logarithmically divergent subdiagrams, but it has an overall logarithmic divergence. The Λ-dependence of this contribution to the free energy can only be cancelled by that of the short-distance coefficient f:

$$\Lambda\frac{d}{d\Lambda}f \;=\; -\frac{\lambda^3}{48}\Lambda\frac{d}{d\Lambda}\int_{pq_1q_2q_3}\frac{1}{q_1^2(\mathbf{p}+\mathbf{q}_1)^2 q_2^2(\mathbf{p}+\mathbf{q}_2)^2 q_3^2(\mathbf{p}+\mathbf{q}_3)^2}. \tag{C.5}$$

After combining pairs of propagators using the Feynman parameter trick, the integrals over q_1, q_2, and q_3 can be evaluated analytically using (B.13). The result is

$$\int_{pq_1q_2q_3}\frac{1}{q_1^2(\mathbf{p}+\mathbf{q}_1)^2 q_2^2(\mathbf{p}+\mathbf{q}_2)^2 q_3^2(\mathbf{p}+\mathbf{q}_3)^2}$$

$$= \left[\frac{1}{8\pi}e^{\gamma\epsilon}\mu^{2\epsilon}\frac{\Gamma(\frac{1}{2}+\epsilon)}{\Gamma(\frac{1}{2})}\int_0^1 dx(x-x^2)^{-\frac{1}{2}-\epsilon}\right]^3 \int_p p^{-3-6\epsilon} \tag{C.6}$$

The integral over p vanishes in dimensional regularization if the same scale μ is used in the regularization of ultraviolet and infrared divergences. If a different scale Λ is used for ultraviolet divergences, the value of the integral is

$$\int_{pq_1q_2q_3} \frac{1}{q_1^2(\mathbf{p}+\mathbf{q}_1)^2 q_2^2(\mathbf{p}+\mathbf{q}_2)^2 q_3^2(\mathbf{p}+\mathbf{q}_3)^2} = \frac{1}{32(16\pi)^2}\left[\left(\frac{1}{\epsilon}\right)_{UV} - \left(\frac{1}{\epsilon}\right)_{IR} + 8\log\frac{\Lambda}{\mu}\right], \tag{C.7}$$

Inserting this result into (C.6), we obtain the evolution equation

$$\Lambda\frac{d}{d\Lambda}f = -\frac{\pi}{12}\left(\frac{\lambda}{16\pi}\right)^3. \tag{C.8}$$

This result is accurate to all orders in λ and to leading order in the coefficients of higher dimension operators.

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Figure Captions

1. Feynman diagrams for the logarithm of the partition function in the full theory and in the effective theory.

2. Additional Feynman diagrams for the logarithm of the partition function in the effective theory.

3. Feynman diagrams for the self-energy in the full theory and in the effective theory.

4. Additional Feynman diagram for the self-energy in the effective theory.

5. Four-loop diagrams for the logarithm of the partition function of the effective theory which have logarithmic ultraviolet divergences.

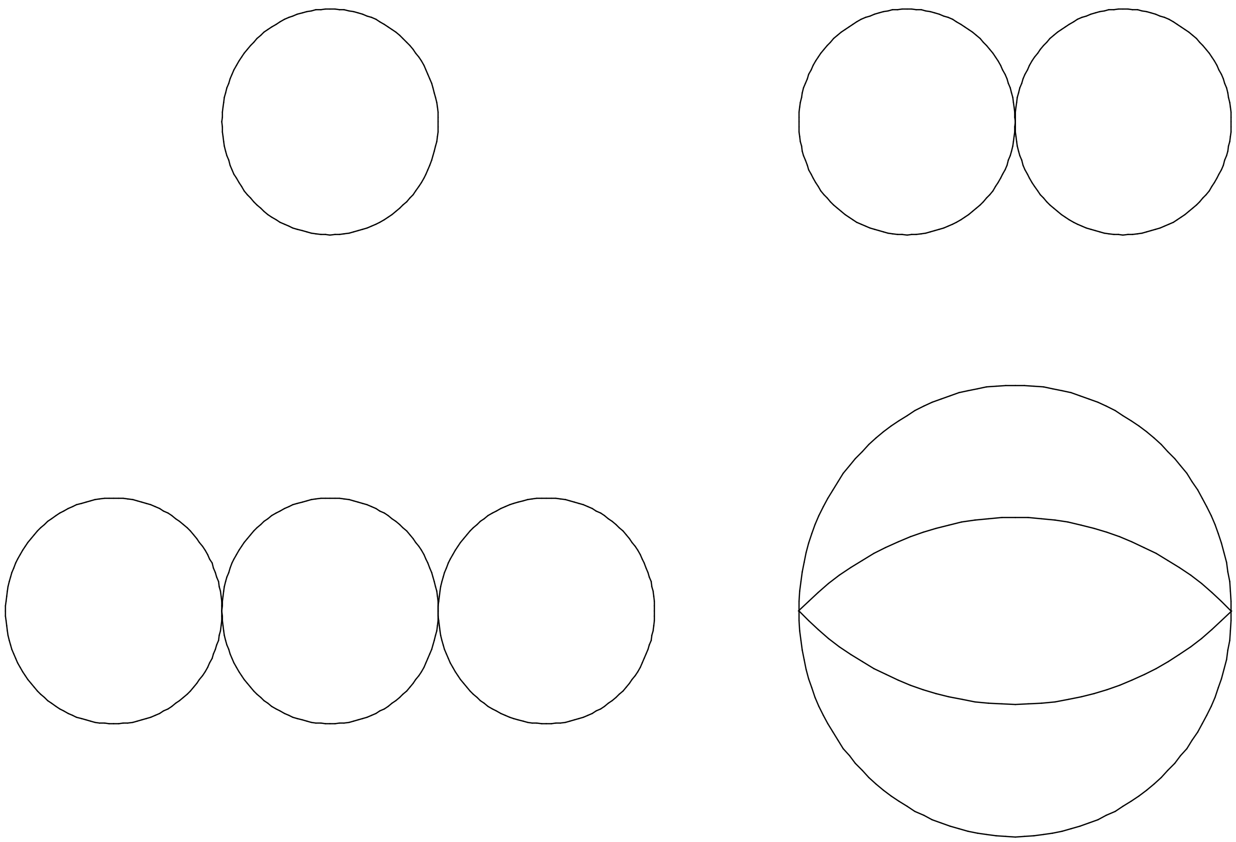

Figure 1

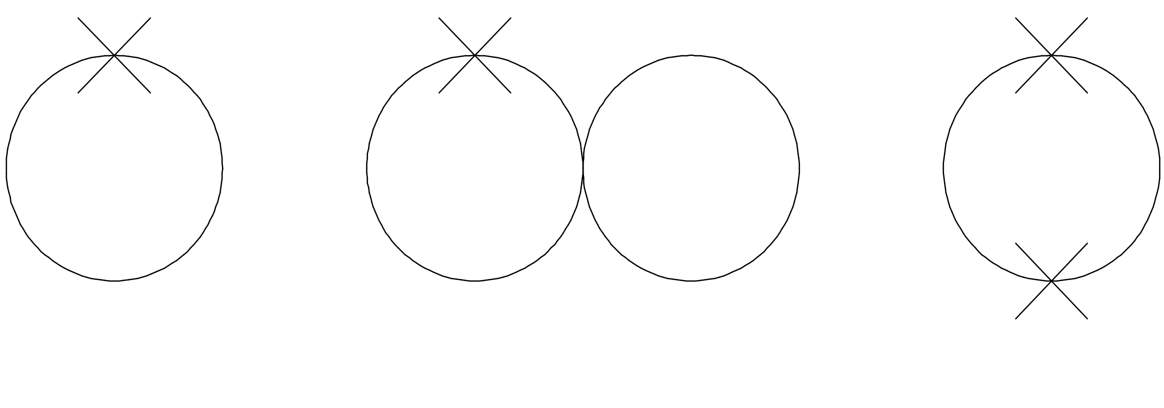

Figure 2

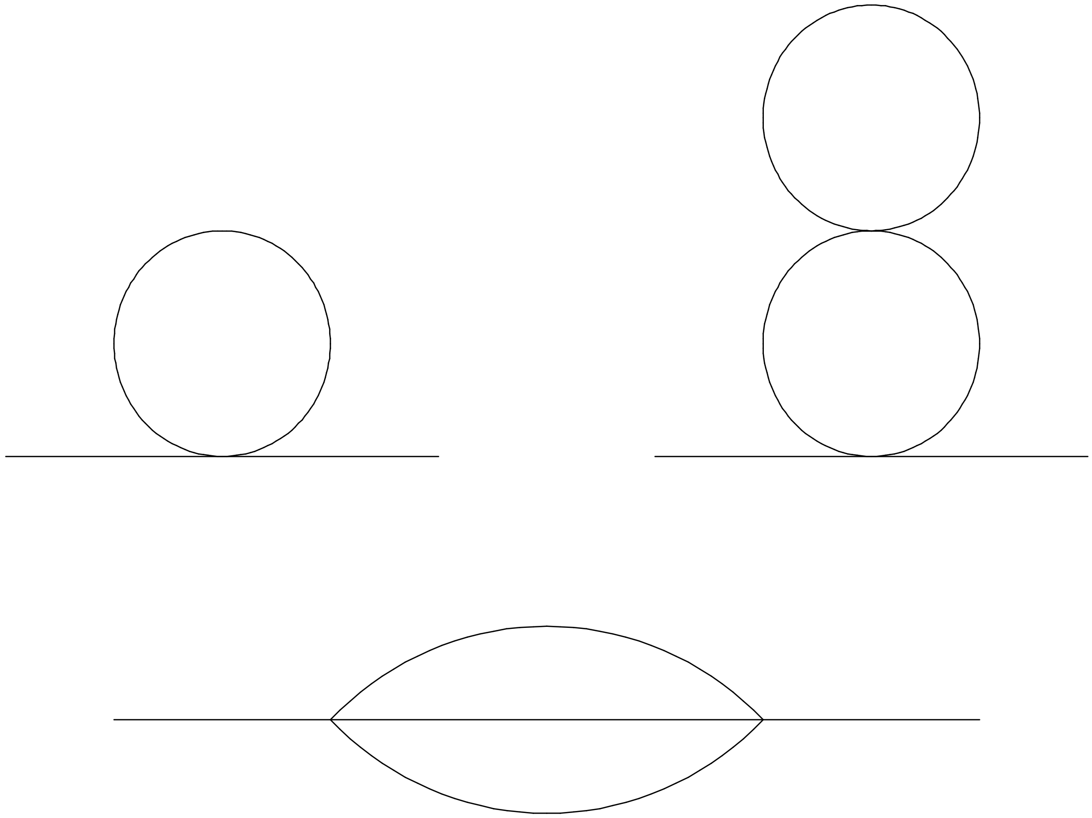

Figure 3

Figure 4

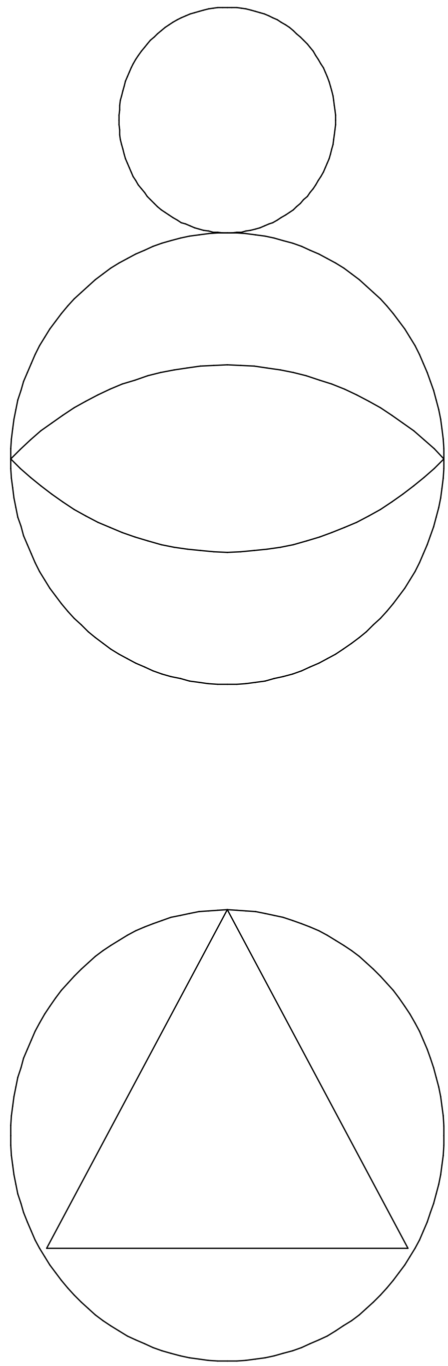

Figure 5